

More Examples

Often $\text{curl}(\vec{F})$ is simpler than curl

Ex: Compute $\iint_S \text{Curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and S the part of sphere $x^2 + y^2 + z^2 = 4$ inside cylinder $x^2 + y^2 = 1$ and above xy -plane.

Solution 1: Compute with Stokes

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y) \langle 1, 1, 0 \rangle$$

$$\vec{S}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), \sqrt{4-r^2} \rangle \text{ or } (r, \theta) \in$$

$$[0, 1] \times [0, 2\pi]$$

$$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -\frac{1}{2}(4-r^2)^{-1/2} \cdot 2r \rangle = \langle \cos(\theta), \sin(\theta), -r(4-r^2)^{-1/2} \rangle$$

$$\vec{S}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$$

$$\vec{S}_r \times \vec{S}_\theta = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -r(4-r^2)^{-1/2} \\ -r \sin(\theta) & r \cos(\theta) & 0 \end{vmatrix} = \langle r^2(4-r^2)^{-1/2} \cos \theta, r^2(4-r^2)^{-1/2} \sin \theta, r \rangle$$

$$\begin{aligned} \therefore \iint_S \text{Curl}(\vec{F}) dS &= \iint_D \text{Curl}(\vec{F})(\vec{S}(r, \theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) dA \\ &= \iint_D (r \cos(\theta) - r \sin(\theta)) \langle 1, 1, 0 \rangle \cdot \langle r^2(4-r^2)^{-1/2} \cos \theta, r^2(4-r^2)^{-1/2} \sin \theta, r \rangle \\ &= \iint_D r \cdot r^2 (4-r^2)^{-1/2} (\cos^2(\theta) - \sin^2(\theta)) dA \\ &= \int_{\theta=0}^{2\pi} \cos 2\theta \int_0^1 r \cdot r^2 (4-r^2)^{-1/2} dr d\theta \\ &= -\frac{1}{2} \int_{\theta=0}^{2\pi} \cos(2\theta) \int_{u=4}^3 (4-u) u^{-1/2} du d\theta \\ &= -\frac{1}{2} \int_{\theta=0}^{2\pi} \cos(\theta) \left[8u^{1/2} - \frac{2}{3}u^{3/2} \right]_4^3 d\theta = \left(\frac{16}{3} - 8\sqrt{3} \right) \cdot (0) = \boxed{0} \end{aligned}$$

Sol 2: (with Stokes's Theorem).

$$\iint_S \text{Curl}(\vec{F}) dS = \int_{\partial S} \vec{F} \cdot d\vec{r} \Rightarrow \text{parametric } dS \text{ via}$$

$$\vec{F}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3} \rangle$$

$$\vec{F}'(\theta) = \langle -\sin(\theta), \cos(\theta), 0 \rangle$$

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$$\therefore \iint_S \text{Curl}(\vec{F}) \cdot d\vec{s} = \int_0^{2\pi} \vec{F}'(\theta) \cdot \vec{r}'(\theta) d\theta = \int_0^{2\pi} 0 d\theta = \boxed{0}$$

Note: ① often $\text{Curl}(\vec{F})$ is simpler than \vec{F}

② The Stokes Equation also implies $\iint_S \text{curl}(\vec{F}) \cdot d\vec{s} = \iint_T \text{Curl}(\vec{F}) \cdot d\vec{s} \Rightarrow (\partial S = \partial T)$

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle xy, yz, zx \rangle$ and C the boundary of the part of $z = 1 - x^2 - y^2$ in the first octant.

Sol: Note that C has three pieces.

Solution: first, try Stokes's Theorem.

$$\vec{S}(r, \theta) = \langle r \cos(\theta), r \sin(\theta), 1 - r^2 \rangle \text{ or } (r, \theta) \in [0, 1] \times [0, \frac{\pi}{2}]$$

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \langle y, z, x \rangle$$

$$\therefore \text{Curl}(\vec{F})(\vec{S}(r, \theta)) = -\langle \sin(\theta), 1 - r^2, \cos(\theta) \rangle$$

$$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -2r \rangle \quad \vec{S}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$$

$$\vec{S}_r \times \vec{S}_\theta = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta & \sin \theta & -2r \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} = \langle 2r^2 \cos \theta, 2r^2 \sin \theta, r \cos^2 \theta + r \sin^2 \theta \rangle$$

$$\text{Sol: } \therefore \text{Curl}(\vec{F})(\vec{S}(r, \theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) = -r(2r^2 \sin(\theta) \cos(\theta) + 2(1 - r^2) r \sin(\theta) + r \cos(\theta))$$

$$\begin{aligned} \therefore \int_C \vec{F} \cdot d\vec{r} &= \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{Curl}(\vec{F}) \cdot d\vec{s} \\ &= \iint_D \text{Curl}(\vec{F})(\vec{S}(r, \theta)) \cdot (\vec{S}_r \times \vec{S}_\theta) dA \\ &= \int_0^1 \int_0^{\pi/2} -r^2(r \sin(2\theta) + 2(1 - r^2) \sin(\theta) + \cos(\theta)) d\theta \end{aligned}$$

$$\begin{aligned} &= \int_0^1 -r^2 (r - 2(1-r^2) + 1) dr = \int_0^1 -r^2 (2r^2 + r - 1) dr \\ &= \int_0^1 -2r^4 - r^3 + r^2 dr \\ &= \left[-\frac{2}{5} - \frac{1}{4} - 1 \right] \end{aligned}$$